

Exercise 55

Differentiate each trigonometric identity to obtain a new (or familiar) identity.

$$(a) \tan x = \frac{\sin x}{\cos x} \qquad (b) \sec x = \frac{1}{\cos x}$$

$$(c) \sin x + \cos x = \frac{1 + \cot x}{\csc x}$$

Solution

Differentiate both sides of each identity.

Part (a)

$$\begin{aligned} \frac{d}{dx}(\tan x) &= \frac{d}{dx} \left(\frac{\sin x}{\cos x} \right) \\ \sec^2 x &= \frac{\left[\frac{d}{dx}(\sin x) \right] \cos x - \left[\frac{d}{dx}(\cos x) \right] \sin x}{\cos^2 x} \\ \frac{1}{\cos^2 x} &= \frac{(\cos x) \cos x - (-\sin x) \sin x}{\cos^2 x} \\ &= \frac{\cos^2 x + \sin^2 x}{\cos^2 x} \end{aligned}$$

Multiply both sides by $\cos^2 x$.

$$1 = \cos^2 x + \sin^2 x$$

Part (b)

$$\begin{aligned} \frac{d}{dx}(\sec x) &= \frac{d}{dx} \left(\frac{1}{\cos x} \right) \\ \sec x \tan x &= \frac{\left[\frac{d}{dx}(1) \right] \cos x - \left[\frac{d}{dx}(\cos x) \right] (1)}{\cos^2 x} \\ \frac{\tan x}{\cos x} &= \frac{(0) \cos x - (-\sin x)(1)}{\cos^2 x} \\ &= \frac{\sin x}{\cos^2 x} \end{aligned}$$

Multiply both sides by $\cos x$.

$$\tan x = \frac{\sin x}{\cos x}$$

Part (c)

$$\begin{aligned} \frac{d}{dx}(\sin x + \cos x) &= \frac{d}{dx} \left(\frac{1 + \cot x}{\csc x} \right) \\ \frac{d}{dx}(\sin x) + \frac{d}{dx}(\cos x) &= \frac{\left[\frac{d}{dx}(1 + \cot x) \right] (\csc x) - \left[\frac{d}{dx}(\csc x) \right] (1 + \cot x)}{\csc^2 x} \\ (\cos x) + (-\sin x) &= \frac{(-\csc^2 x)(\csc x) - (-\csc x \cot x)(1 + \cot x)}{\csc^2 x} \\ \cos x - \sin x &= \frac{-\csc^3 x + \csc x \cot x + \csc x \cot^2 x}{\csc^2 x} \\ &= -\csc x + \frac{\cot x}{\csc x} + \frac{\cot^2 x}{\csc x} \\ &= -\csc x + \frac{\frac{\cos x}{\sin x}}{\frac{1}{\sin x}} + \frac{\cot^2 x}{\csc x} \\ &= -\csc x + \cos x + \frac{\cot^2 x}{\csc x} \end{aligned}$$

Subtract $\cos x$ from both sides.

$$\begin{aligned} -\sin x &= -\csc x + \frac{\cot^2 x}{\csc x} \\ -\frac{1}{\csc x} &= -\csc x + \frac{\cot^2 x}{\csc x} \end{aligned}$$

Multiply both sides by $-\csc x$.

$$1 = \csc^2 x - \cot^2 x$$

Bring $\cot^2 x$ to the left side.

$$\cot^2 x + 1 = \csc^2 x$$